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REPRESENTATION AND TRANSFER OF ABSTRACT MATHEMATICAL CONCEPTS IN ADOLESCENCE AND YOUNG ADULTHOOD

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By adolescence, students are learning more abstract and complex concepts, such as those of algebra and geometry. It is tempting to introduce these concepts through concrete, familiar instantiations that might deeply engage students in the learning process and possibly facilitate initial learning. However, a primary goal of acquiring mathematical concepts is the ability to apply structural knowledge outside the learning situation, and there is evidence that concrete instantiations can hinder transfer. This chapter addresses how successful analogical transfer is influenced by characteristics of the learning and target domains. We discuss results of a series of studies demonstrating that learners are more able to transfer mathematical structure from a learned generic instantiation than from a learned concrete instantiation. We suggest that concrete instantiations of abstract concepts communicate more extraneous information than their more abstract, generic counterparts. This extraneous information is retained in the learner's representation of the concept and hinders subsequent transfer. Implications for learning abstract concepts such as mathematical concepts in adolescence and young adulthood are discussed.

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The period from adolescence to young adulthood is a time when mathematical reasoning and problem solving become more sophisticated. In the preschool and elementary school years, much of children's mathematical knowledge concerns numbers and arithmetic. By adolescence, children are acquiring more abstract and complex concepts, such as those of algebra and geometry. For most students, the acquisition of this knowledge is not without its difficulties. How should mathematical concepts, such as probability theory, exponential growth, and rates of change, be introduced to students to ease these difficulties and best promote their acquisition and application to real-world problems? This chapter addresses how successful analogical transfer is influenced by characteristics of the learning and target domains. In particular, we discuss concepts featured in high school and college curricula, such as exponential growth, group theory, and rates of change (e.g., in physics).

One possibility is that such concepts are well acquired through concrete instantiations such as contextualized, real-world examples. Concrete approaches to learning have been advocated not only for very young children but also for older learners, such as adolescents and adults (for a review, see Anderson, Reder, & Simon, 1996). Support for such approaches often stems from the belief that because cognition is bound to specific situations, teaching abstractions is ineffective. Alternatively, students might learn more effectively through more abstract, generic instantiations of mathematics that present a minimal amount of extraneous information, such as traditional mathematical notation involving generic symbols not tied to specific situations. For example, *acceleration* is defined as the rate of change of velocity with respect to time. Students could learn the concept of acceleration through a concrete context of gravitational acceleration affecting falling objects or instead through the generic expression of $a = \frac{\Delta v}{\Delta t}$ where a is acceleration, Δv is change in velocity, and Δt is change in time.

To evaluate the effectiveness of concrete and generic instantiations, several questions should be considered. What is the definition of a concrete instantiation? What constitutes successful acquisition of a mathematical concept? How does learning a particular instantiation shape the internal representation of a mathematical concept and influence the learner's ability to transfer mathematical knowledge to novel analogous situations? In this chapter, we discuss the results of a series of studies conducted to address these questions. We begin by presenting an interpretation of concrete and abstract instantiations of mathematical concepts and an overview of previous findings on analogical transfer.

CONCRETENESS

In everyday practice, the term *concrete* is typically used in contrast to *abstract* often to differentiate what can and cannot be directly experienced by the senses. These terms can be used in seemingly different situations. For example, there would be little disagreement that the concept "cat" is more concrete than the concept "infinity." This is a comparison of the concreteness of two different concepts. Concreteness can also be compared between instantiations of the same concept; there would also be little disagreement that a real cat is a more concrete instantiation of the category "cat" than a schematic outline. Do these examples point to the same way of defining concreteness across different situations? We suggest that the answer is yes. In both cases, concreteness could be measured by the amount of information (or the amount of entropy reduction) communicated by a given concept or instantiation. Under this view, the concept "cat" communicates the presence of a feline animal and all the known facts associated with cats, assuming that one has prior knowledge of cats. The concept "infinity" communicates much less information (in fact, any set could potentially be infinite), thus leaving a great deal of uncertainty. Similarly, a real cat leaves less uncertainty than a schematic outline that does not communicate information such as color, size, or age. In both cases, therefore, the former is substantially more concrete than the latter.

For instantiations of a fixed concept such as "cat," if concreteness can be measured by the amount of communicated information, then *concrete* and *abstract* are not dichotomous; rather, they lie on a continuum over which the amount of communicated information varies. Specifically, for a given concept, instantiation *A* is more concrete than instantiation *B*, if *A* communicates more information than *B*. Furthermore, an instantiation of a concept (e.g., a particular cat) is often represented by a symbol that communicates information either perceptually, by the amount of detail in the physical stimuli, or verbally, by providing descriptions with different amounts of detail. Perceptually communicated concreteness often results in greater perceptual richness of an instance, which could be measured by physical properties such as contrast and spatial frequency.

To elaborate this point, consider the concept of "person" and how possible symbols can communicate different degrees of information. For example, images in Figure 3.1 communicate increasing amounts of information from left to right. Little could be said with certainty about the leftmost instantiation of a person; this most abstract, generic instantiation communicates only numerosity—the fact that there is a single individual. On the other hand, much could be said about the rightmost instantiation. Namely, this is a specific person, she is a young female, and she was born to an Asian parent. There would be even more information that could be retrieved from memory if the photograph depicts someone you know.

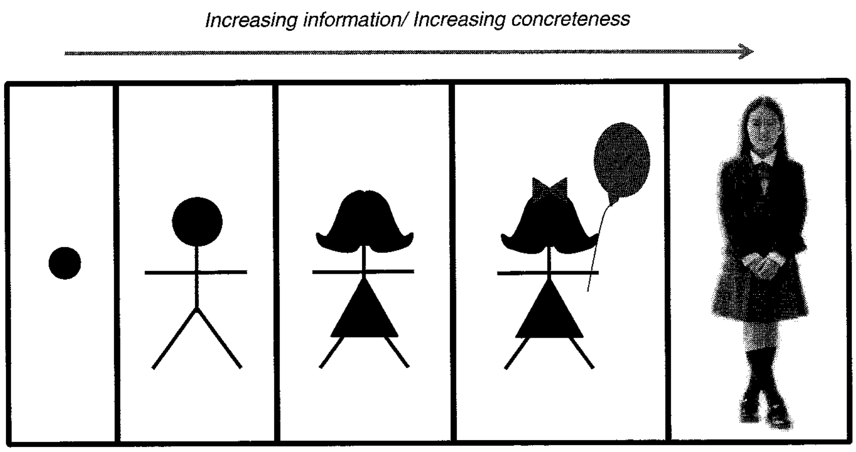


Figure 3.1. Possible symbols for the concept *person.*

It seems that a more abstract instantiation is often a better symbol of a given concept, to denote the entire group, than a more concrete instantiation. This is because a more concrete instantiation communicates much more information, part of which could be extraneous to the concept in question. For example, a stick figure can symbolize any person, yet a young schoolgirl may not well symbolize any person because it communicates additional information such as age and gender, which is nonessential to the concept of *person.* Similarly, a concrete instantiation may be a poor symbol for another concrete instantiation that does not share the same superficial features. Therefore, a schoolgirl is not a good symbol for a middle-aged man, and a middle-aged man is not a good symbol for a schoolgirl.

By the same reasoning, generic instantiations may serve as better symbols when attempting to communicate new, to-be-learned information about a given concept. For example, the lifetime risk of heart disease regardless of gender would probably be better communicated with a stick figure or generic example of the general population than with a picture of a young girl. Similarly, a photograph of a young girl would not make a good symbol when communicating information about a specific subset of the general population, such as the prevalence rate for prostate cancer in middle-aged men.

NATURE OF MATHEMATICAL CONCEPTS AND THEIR INSTANTIATIONS

Just as many everyday object concepts and their instantiations can vary in the amount of communicated information, mathematical concepts and their instantiations can also vary in the degree of communicated information.

However, there are critical differences between mathematical concepts and concepts such as "cat" or "person." Most everyday concepts are ill-defined (see Solomon, Medin, & Lynch, 1999) in the sense that their definitions can vary across cultures, individuals, and time. Furthermore, everyday concepts such as "cat" are grounded in perceptual similarity and acquired with little effort through encounters with instances of the concept (Kloos & Sloutsky, 2008). For example, most cats tend to have common observable features—similar size, four legs, whiskers, and pointed ears—and young children acquire this concept easily. Mathematical concepts, however, have precise definitions based on their relational structure. For example, exponential growth is defined as the change in quantity N according to the following formula:

$$N(t) = N_0 e^{\alpha t} \tag{1}$$

for a variable t and constant α, where e^x is the exponential function and N_0 is the initial value of N. Therefore, the concept of exponential growth is defined by the relational pattern between N_0, $N(1)$, $N(2)$, etc. Instances of mathematical concepts specify additional information beyond the defining relational structure. Instantiations of exponential growth would specify particular values of the constants α and N_0. More concrete instantiations would convey more information—perhaps growth of a particular population of wild eastern cottontail rabbits living in the midwestern United States.

Therefore, for mathematical concepts, instances can be vastly dissimilar, sharing few directly observable similarities. For example, in addition to describing populations of rabbits, exponential growth/decay can describe the metabolism of medication in the body and the temperature of a cooling cup of coffee. Because superficial features can vary widely, it is often difficult to spontaneously recognize instances of the same concept. As a result, the acquisition of such concepts is often difficult for both children and adults and typically requires some supervision (e.g., Kloos & Sloutsky, 2008), which may take the form of explicit instruction that begins with an initial instantiation.

One goal of learning mathematics is the ability to appropriately apply mathematical knowledge to novel situations. Therefore, an effective instantiation must promote two processes: learning of the instantiation and transfer of defining relational structure to a novel situation that is structurally analogous, or isomorphic. For example, successfully acquiring the concept of exponential growth from learning about growth of a rabbit population would imply that knowledge of exponential growth would be recognized and applied to (at least some) novel analogous situations, such as monetary growth of investments.

ANALOGICAL TRANSFER

How likely is it that structural knowledge will transfer outside of the learned situation? The past 20 years have produced a consensus on some aspects of analogical transfer. First, spontaneous analogical transfer is notoriously poor. This finding has been documented in numerous studies with both adults and children (e.g., Gick & Holyoak, 1980, 1983; Goswami, 1991; Novick, 1988; Reed, Dempster, & Ettinger, 1985; Reed, Ernst, & Banerji, 1974; Simon & Reed, 1976). Second, a factor that mediates transfer is similarity of the base and target domains. Transfer to similar instances, or near transfer, is more likely to occur than transfer to dissimilar instances, or far transfer (Holyoak & Koh, 1987; Holyoak & Thagard, 1997; Ross, 1987, 1989). High surface similarity between the base and the target domain can facilitate spontaneous retrieval of prior knowledge (Gentner, Ratterman, & Forbus, 1993). For example, college students who learned solutions to probability story problems were more likely to remember solution strategies and formulas when presented with novel isomorphic problems that involved similar story lines (e.g., both study and test problems involved mechanics randomly choosing cars to work on) rather than dissimilar story lines (e.g., the study problem involved mechanics choosing cars, and the test problem involved scientists choosing computers; Ross, 1987, 1989).

Finally, there is evidence that during successful analogical transfer, the reasoner aligns the learned and novel domains according to common structure (Gentner, 1983, 1988; Gentner & Holyoak, 1997; Holyoak & Thagard, 1989). Similarity can also affect transfer by affecting the process of structural alignment. Because similar elements are easier to align than dissimilar ones (Gentner, 1983, 1988; Gentner & Markman, 1997; Markman & Gentner, 1993), structural alignment is facilitated when similar elements play identical structural roles across the learning and transfer domains. As a result, transfer is more successful when similar elements hold analogous roles in both domains (e.g., for probability problems, both study and test problems involve mechanics choosing cars; Ross, 1987, 1989; for related findings, see Reed, 1987). However, when similar elements hold different structural roles across domains (e.g., the study problem involves mechanics choosing cars and the test question involves car owners choosing mechanics), learners tend to misalign structure by matching common elements (Ross, 1987, 1989), and consequently transfer fails.

Although surface features can affect both recall of previous domains and alignment between two domains, there is evidence that they can also influence the manner in which learners interpret the structure of a domain. In a series of studies involving algebra word problems, Bassok and her colleagues have demonstrated that students often interpret structure through the context in which it is presented (for summaries, see Bassok, 1996, 2003). When undergraduate students with no prior knowledge of probability theory were asked to solve permutation problems, their spontaneous solutions typically reflected semantic symmetry or asymmetry of the elements of the problem (Bassok, Wu, & Olseth, 1995). For example, some problems involved m secretaries assigned n computers. In everyday scenarios, secretaries and computers often play asymmetric semantic roles because they are different types of entities and secretaries may use computers. Participants generally placed them in asymmetric arithmetic roles, often involving m in the numerator and n in the denominator (e.g., $\frac{[m^3]}{[n!]}$). Students tended to generate categorically different solutions to isomorphic problems involving elements that are interpreted as semantically symmetric. For instance, when given problems involving m doctors working with n doctors, participants tended to place m and n in structurally symmetric roles (e.g., $\frac{[m+n]}{[mn]^3}$, here m and n appearing in both the numerator and denominator).

The behavior of interpreting structure through semantics of the context can often be a smart approach to problem solving because mathematics is often used to model the structure of real-world situations and therefore semantic structure often correlates with mathematical structure. For example, it is probably more expected to add a number of roses and a number of tulips than to divide a number of roses by a number of tulips. It is also probably more expected to divide a number of roses by a number of vases than to add a number of roses to a number of vases. The downside of using context to interpret structure occurs when attempting to transfer between two isomorphs that do not share a common structural interpretation. For example, when students learned solutions to permutation problems in a semantically asymmetric context (e.g., tulips to vases), they successfully transferred solution strategies to novel asymmetric problems but failed to do so to symmetric problems (e.g., tulips and roses; Bassok, Wu, & Olseth, 1995). Not only did participants fail to transfer, they were very confident that the two problems differed in their mathematical structure.

Transfer failure attributed to different structural interpretations has also been demonstrated between continuous and discrete models of change (Bassok & Olseth, 1995). For example, the change in the volume of water in a pool would be continuous (able to take on any real number within some range of numbers), whereas the change in people in a pool would be discrete (limited to only a subset of values within a range of numbers, in this case whole numbers). In one study, undergraduate students were taught solutions to word problems and then given novel problems in a different context. All the problems involved constant rates of change and could be solved by the same solution strategy. The contexts included populations, money, and basic physics. What differed between base and target domains was not only the cover story but also whether the change was interpreted as continuous or discrete. For example, some contexts involved the rate at which ice is melting from a glacier, while other contexts involved the rate at which ice is regularly delivered to a restaurant. When both base and target domains shared the type of change (continuous or discrete), transfer was much more likely than when the domains differed in type of change. Furthermore, an asymmetry was found in which transfer was more likely to occur from a discrete-change domain to a continuous-change domain than the reverse. These findings demonstrate that learners often interpret structure through context and that their interpretations can lead to transfer failure when novel isomorphs have contexts that appear to be structurally different.

One way of facilitating successful transfer from concrete instantiations is through explicit comparison of multiple instances. Several studies involving both children and adults have demonstrated better performance on relational tasks after comparing two instances than after learning only one instance or learning two instances sequentially (e.g., Catrambone & Holyoak, 1989; Gentner, Loewenstein, & Hung, 2007; Gentner, Loewenstein, & Thompson, 2003; Gentner & Namy, 2004; Gick & Holyoak, 1983). Adults who learned negotiation strategies (e.g., compromise on all issues vs. trade-off on specific issues between two parties) were more successful transferring learned strategies to novel situations when they first compared and noted similarities of two examples relative to those who only read and summarized the examples separately (Gentner et al., 2003). There is also some evidence of better conceptual and procedural knowledge of mathematical equation solving after middle school students compared two examples, particularly when the examples presented different solution methods, than after learning examples in succession (Rittle-Johnson, Star, & Durkin, 2009). The process of comparison can highlight common relational structure (Kotovsky & Gentner, 1996) and result in the construction of an abstract schematic representation of knowledge (Catrambone & Holyoak, 1989; Gick & Holyoak, 1983). Schematic knowledge representations can in turn promote subsequent transfer (Gick & Holyoak, 1983; Novick & Holyoak, 1991; Ross & Kennedy, 1990).

Taken together, prior research suggests that learners can form representations of abstract concepts, including mathematical concepts, through learning concrete instantiations, but these representations are far from purely abstract. A purely abstract representation, like mathematical definitions themselves, would contain nothing beyond the structural relations. However, internal representations contain considerable superficial information retained from the learning context. The existence of this information in a representation is not necessarily a bad thing per se. This information may be harmless in the case of a teacher being able to illustrate multiple examples of mathematical models of real-world phenomena. This information may, in some instances, be helpful because it may facilitate transfer to analogous, superficially similar situations. The negative impact occurs when nonessential information is interpreted as essential. The learner incorporates this information into the representation of the concept, and as a result transfer fails when potential transfer domains lack this extraneous information. Learning and comparing multiple instances can highlight common relational structure. The highlighting of common relations likely lessens the representational weight of any one set of superficial features and as a result a schematic representation is formed. However, an abstract schema does not appear to supplant mental representations of individual exemplars. As Medin and Ross (1989) suggested, abstract and specific knowledge coexist, with reasoning often case-based and induction often conservative.

SUPPORT FOR THE USE OF CONCRETE MATERIAL IN TEACHING

The previously discussed studies have investigated analogical transfer from a variety of concrete instantiations. In educational practice, the use of concrete instantiations to present mathematics is widespread. Several arguments support this practice (for discussions, see McNeil & Uttal, 2009; Uttal, Scudder, & DeLoache, 1997). First, some developmental theories posit that development proceeds from the concrete to the abstract (e.g., Bruner, 1966; Montessori, 1917; Piaget, 1970), and therefore teaching and learning should follow the same sequence (for a discussion, see McNeil & Uttal, 2009). Second, concrete instantiations may be more engaging for the learner than more abstract, generic instantiations; certainly, engagement in learning is necessary. Third, some concrete instantiations may tap prior knowledge and therefore facilitate initial learning.

There is some evidence that mathematical problem solving can be more accurate when presented in familiar, concrete contexts than when presented as decontextualized, symbolic mathematics. For example, adolescent Brazilian street vendors were able to solve arithmetic problems in the contexts of their sales but were unable to solve the same problems presented as symbolic mathematics (Carraher, Carraher, & Schliemann, 1985). Yet evidence of the effectiveness of concrete instantiations in teaching formal mathematics is not unequivocal (Sowell, 1989; Uttal, Liu, & DeLoache, 1999; Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009). For example, Koedinger and Nathan (2004) demonstrated that algebra students were more successful in solving simple story problems than analogous mathematical equations, often using informal strategies such as guess-and-check to arrive at accurate solutions. However, for more complex problems, the reverse was the case: Students were more successful solving symbolic equations than solving word problems (Koedinger, Nathan, & Alibali, 2008). Therefore, concrete contexts may *sometimes* provide an advantage over decontextualized symbolic mathematics for problem solving. It is important to note that these findings were demonstrated for learning and problem solving in a single context (that of the individual problem). Nevertheless, for mathematical concepts, an important goal of learning is not only to acquire knowledge and problem-solving ability in a particular context but also to transfer the acquired mathematical knowledge to multiple novel contexts. Thus, while students may sometimes more accurately solve problems with concrete instantiations than analogous symbolic instantiations, the question remains: How likely is it that students will transfer the mathematical structure learned from concrete instantiations versus one learned from generic instantiations?

CONCRETENESS AS PERCEPTUAL RICHNESS

As discussed earlier, one dimension of concreteness is perceptual richness. Perceptual richness can hinder transfer of relations for both children and adults. One line of evidence comes from studies of young children's symbol use (DeLoache, 1991, 2000). Successful symbol use requires transfer of relations from one domain to another. For example, to effectively use a map as a symbol for a real location, one must recognize the common relations between entities on the map and their real-world analogs. In one study, children ages 2 and 3 years were shown the location of a toy in either a three-dimensional scale model or a two-dimensional picture and then asked to retrieve a real toy in an analogous location in a real room. Perhaps counterintuitively, those who were shown the picture were more successful than those who were shown the more realistic concrete model. A similar advantage for more generic material was found for prelinguistic infants, who were better able to extend labels from generic, perceptually sparse objects to perceptually rich objects than the reverse (Son, Smith, & Goldstone, 2008). Clearly, if young children benefit from generic instantiations of concepts, adolescents and young adults could be expected to do so, too.

As expected, not only does perceptual richness hinder young children's ability to transfer simple relations, it also can hinder adults' ability to transfer acquired knowledge of more complex structures. In one study (Goldstone & Sakamoto, 2003), undergraduate students learned the principle of competitive specialization, which explains how individual agents self-organize without a central plan. When students learned through a scenario of ants foraging for food, transfer to a novel isomorph was more successful when the ants and food were depicted more abstractly as dots and patches than when the depictions resembled ants and apples.

The studies discussed thus far varied the perceptual richness of the same instantiation. In contrast, we wanted to investigate the effect of concreteness, including perceptual richness, varied across different instantiations of the same mathematical structure. This is analogous to real-world scenarios in which mathematics may transfer between instantiations of different degrees of concreteness, such as generic symbolic notation and perceptually rich, scientific applications. In a series of studies, we varied the concreteness of the learning instantiation to consider its effect on transfer of mathematical structure. We chose a simple mathematical concept that we could instantiate in a variety of different ways that would appear novel to our study participants. The concept was that of a commutative mathematical group of order 3. This is a set of three elements, or equivalence classes, and an associated operation that has the properties of associativity and commutativity. In addition, the group has an identity element and inverses for each element (see Table 3.1 for properties). In our experiments, training was presented via computer and consisted of explicit presentation of the group rules using the elements of the given instantiation, questions with feedback, and examples. After training, participants received a multiple-choice test of novel complex questions.

In our first experiment, we considered concreteness as perceptual richness of the elements and context. Undergraduate students were trained and tested with an abstract, generic instantiation and a perceptually rich, concrete instantiation (Sloutsky, Kaminski, & Heckler, 2005). The generic instantiation was described as a written language involving three simple, monochromatic symbols in which combinations of two or more symbols yield a predictable resulting symbol. Statements were expressed as *symbol 1, symbol 2 → resulting symbol*. The concrete condition presented an artificial phenomenon involving images of three colorful, three-dimensional shapes. Participants watched movies of two or more shapes coming into contact, then

TABLE 3.1
Principles of Commutative Mathematical Group

A commutative group of order 3 is a closed set of three elements and a binary operation (denoted +) with the following properties:	
Associative	For any elements x, y, z: $((x + y) + z) = (x + (y + z))$
Commutative	For any elements x, y: $x + y = y + x$
Identity	There is an element, $\mathbf{I}$, such that for any element, x: $x + \mathbf{I} = x$
Inverses	For any element, x, there exists another element, y, such that $x + y = \mathbf{I}$

disappearing, and a resulting shape appearing. For both instantiations, the resulting symbol or shape was specified by the mathematical structure. After training and testing of one instantiation, participants were trained and tested with the other instantiation. We found that participants successfully learned both instantiations, with no difference in mean test score on the generic instantiation no matter which instantiation they learned first. However, there was a marked difference in mean test score on the concrete instantiation, with participants who were initially trained with the generic instantiation scoring higher on the concrete test than did participants who were initially trained with the concrete instantiation. In other words, learning the concrete instantiation resulted in no improved learning of the generic instantiation. On the other hand, learning the generic instantiation resulted in better performance on the concrete instantiation, suggesting that participants were able to transfer their knowledge from the generic to the concrete instantiation.

In a second experiment, we considered the effects of perceptual richness on initial learning. Participants learned an instantiation of a group that had different levels of concreteness: (a) generic black symbols; (b) colorful, patterned symbols; (c) classes of colorful, patterned symbols; or (d) classes of real objects. After training, participants were given a test of novel questions on the same instantiation. While all participants learned the rules, those who learned with the generic symbols scored significantly higher than did the other participants, with no differences across these three conditions (Sloutsky et al., 2005). Therefore, the mere addition of patterns and color lowered learning. Similar negative effects of perceptual richness were demonstrated in another recent study: Children ages 10 to 12 years made more errors on word problems involving money when they were given real bills and coins to help them solve the problems than children who were not given real money (McNeil, Uttal, Jarvin, & Sternberg, 2009).

The results of these experiments indicate that perceptual richness that is irrelevant to the to-be-learned concept hindered both learning and transfer. However, not all concreteness is irrelevant. Some concreteness may help to communicate relevant structure by tapping prior knowledge or by presenting perceptual information that is correlated with structure. This "relevant concreteness" would most likely facilitate learning of a novel concept, but its effect on transfer has not been clear.

RELEVANT CONCRETENESS

To investigate the effects of such relevant concreteness, we instantiated the concept of a mathematical group in a context involving familiar objects that might facilitate learning of the group rules (Kaminski, Sloutsky, & Heckler,

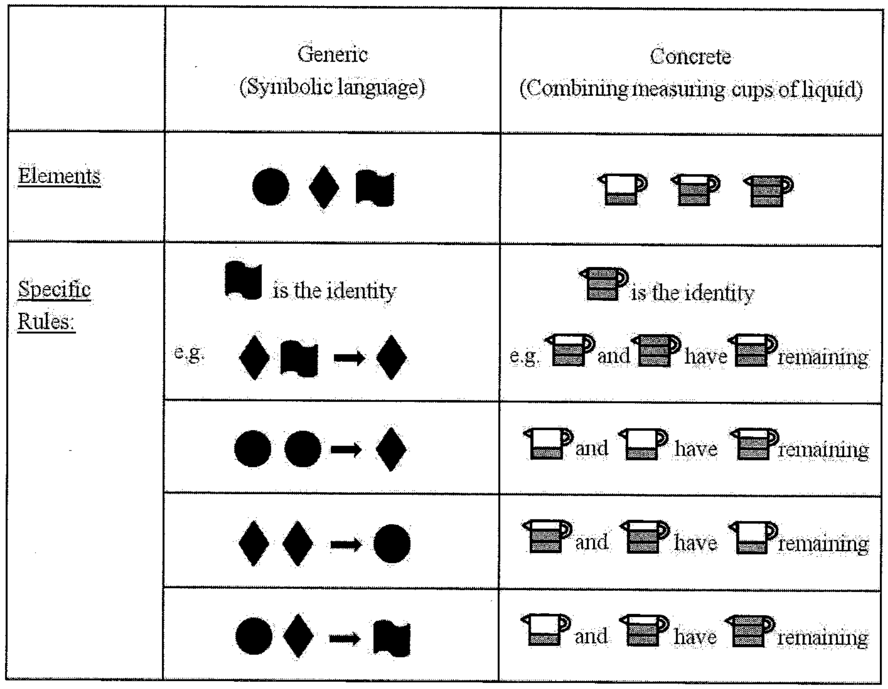
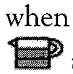
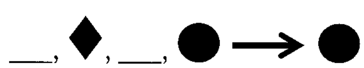


Figure 3.2. Generic and concrete instantiations of a mathematical group.

2005a). In this case, the elements of the group were three measuring cups (see Figure 3.2). Instead of learning arbitrary rules of symbol combinations, participants were told that they needed to determine a leftover amount of liquid when different measuring cups were combined. For example, combining  and  resulted in  left over. We compared learning this instantiation with learning a generic instantiation. This generic instantiation was described, as in our earlier studies, as the rules of a symbolic language. Training consisted of explicit statements of the rules and one example. After training, participants answered a series of multiple-choice questions. The following are example questions from the generic learning condition.

1. What can go in the blanks to make a correct statement?



2. Find the resulting symbol:

The concrete condition presented the analogues of these questions. All training and testing was isomorphic across conditions. Participants in both conditions successfully learned the instantiation, but under the minimal training that they received (only one statement of the rules and one example), the relevantly concrete instantiation did have an advantage over the generic (81% vs. 63% correct, with chance = 38%).

To test whether this advantage would exist for transfer, we gave participants slightly more detailed training, including explicit examples and questions with feedback. Subsequently, as in the previous experiments, participants were tested and then presented with a novel isomorphic instantiation of mathematical group. This novel instantiation was intentionally concrete and contextually rich, as are many real-world instantiations of mathematics, and was described as a children's game from another country. Specifically, participants were asked to figure out the rules of the game. In the game, children point to a series of objects, then the child who is "it" points to a final object. This child wins if he or she points to the correct object according to the rules (see Figure 3.3). Participants were told that the rules of the game were like the rules of the system they had just learned (i.e., either the concrete or the generic instantiation). Then, participants were shown a series of examples from which the rules could be deduced. After seeing the examples, a multiple-choice test, isomorphic to the test of the learning

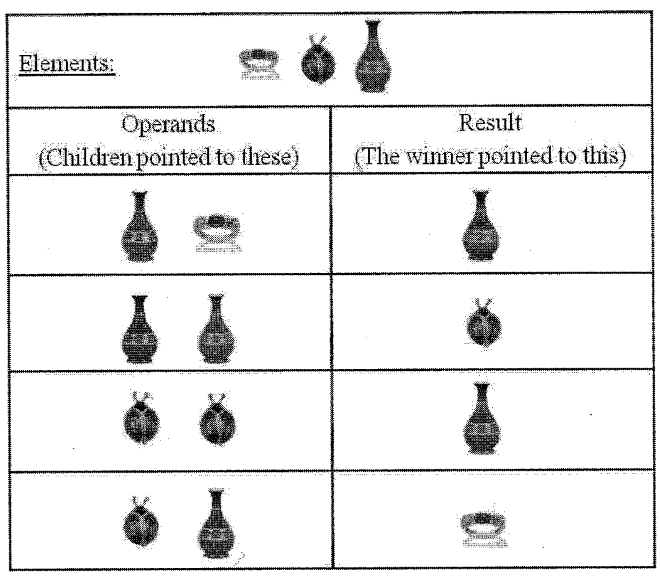
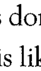


Figure 3.3. Instantiation of a commutative mathematical group used for the transfer domain.

domain, was given. The results revealed that with the slightly protracted training, there was no difference in learning scores across the two conditions (78% correct vs. 75% correct for the concrete and generic conditions, respectively). However, there was a striking difference in transfer. Participants in the concrete condition had an average test score of 54% correct, while the average score in the generic condition was 79% (with chance being 38%).

Because structural alignment is an essential component of successful analogical transfer, we wanted to know whether participants in each condition were able to align structure between the learning and transfer instantiations. As an indication of alignment, participants were asked to match analogous elements across domains. In the generic condition, 100% of participants were able to do so, whereas only 25% of participants in the concrete condition made the correct match. Because there were three elements, we would expect chance performance to result in 33% accuracy (Kaminski, Sloutsky, & Heckler, 2005b). Therefore, those who learned the concrete instantiation scored no better than guessing.

Why were participants in the concrete condition unable to align structure across the learning and transfer domains? There are two possibilities. First, perhaps learners in the concrete condition formed a representation of that instantiation that did not contain the relevant mathematical structure. It is possible that these participants were accurate on the test of the concrete instantiation because the familiar elements and context allowed them to "bootstrap" their way to correct answers without truly acquiring the mathematical structure. This possibility is reminiscent of Koedinger and Nathan's (2004) finding that algebra students often successfully solved simple story problems by using informal strategies without resorting to formal algebraic solutions. The second possibility is that the representation of the concrete instantiation did contain structure, but that structure was tightly tied to the elements and context such that learners were unable to recognize it in novel situations.

To test the possibility that failure to transfer from the concrete instantiation was due to difficulty in aligning structure and not due to failure to represent structure, we conducted another experiment that was identical to the previous one, with a single exception. Prior to the transfer test, we showed half of the participants the matching of analogous elements across domains. In the concrete condition, half were told  is like ,  is like ,  is like . In the generic condition, half were told the analogous alignments between the generic elements and transfer elements. The goal was to assist learners with structural alignment by telling them the correspondence of analogous elements. We found that when learners in the concrete condition were given the correspondence, they transferred as successfully as the learners in the generic condition (85% accuracy for both condition). In the generic condition, there was no significant difference in transfer scores as a function of being given the element correspondence, suggesting that participants were able to spontaneously align structures between the learning and transfer domains (Kaminski, Sloutsky, & Heckler, 2006c). The fact that participants in the concrete condition were successful when assisted with structural alignment also indicates that structure was acquired during learning. If they had not actually learned the mathematical rules, it is highly unlikely that they would perform so well on difficult transfer questions by simply being given a matching of elements.

It seems that when acquiring a novel mathematical concept through a concrete context, structural knowledge is represented but tied to the learning context in a way that inhibits its spontaneous recognition in other situations. To consider this possibility more carefully, we tested whether learned structure could be recognized when instantiated with novel elements. Participants were trained with either the concrete or generic instantiation of the mathematical structure, as in the previous studies. After training, instead of being presented with the transfer domain and a test of complex questions, participants were given a structure discrimination task. On each trial, participants were presented with a set of three expressions. They were told that each set is from a new system and were asked whether the new system followed the same type of rules as the system they had previously learned. Four types of trials were used. Figure 3.4 shows examples of each type, as expressed for the concrete condition. For the generic conditions, the analogous state-

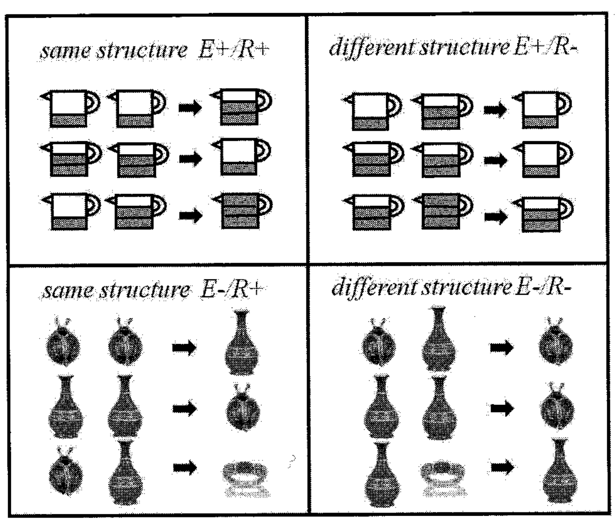
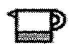


Figure 3.4. Example statements used for the structure discrimination task.

ments were expressed with the generic black symbols. Six trials involved the same elements as the learning phase and the same relational structure (E+/R+). Six trials involved the familiar elements but different relational structure (E+/R–). Six trials involved novel elements and the familiar relational structure (E–/R+). Another six trials involved both novel elements and novel relations (E–/R–). To measure discriminability, we calculated the number of correct "same structure" responses (on R+ trials) minus the number of incorrect "same structure" responses (on R– trials). We measured discriminability separately for familiar elements and novel elements. For familiar elements, participants in both conditions were highly accurate (90% correct). However, there were dramatic differences when it came to novel elements. Accuracy in the generic condition was 78%, while accuracy in the concrete condition was only 26% (Kaminski, Sloutsky, & Heckler, 2006a).

These findings suggest that although structure is represented when learning a concrete instantiation, the most salient aspect of the representation is the superficial, contextual information and not the important structural information. In a follow-up study, we asked participants, after they had learned either the concrete or the generic instantiation, to write down what they recalled about what they had learned. We then counted the number of statements referring to structure such as reiteration of rules (e.g., ♦♦ → ●

or combining  and  has  left over) and the number of statements referring to superficial elements such as "it was about the discovery of a symbolic language" or "it was about liquid in cups." We found that those who learned the generic instantiation made nearly 4 times as many structural comments as those who learned the concrete instantiation. The responses of the participants who learned the concrete instantiation contained approximately twice as many references to the superficial as references to structure. The opposite pattern was observed for participants in the generic condition: They made approximately 3 times as many references to structure as references to the superficial (Kaminski, Sloutsky, & Heckler, 2009). These results support the argument that the representation of the concrete instantiation was overwhelmed by superficial information. They also suggest that structure is salient in the representation of the generic instantiation. These findings parallel ideas of fuzzy trace theory, which posits that transfer succeeds when learners have formed "gist" knowledge representations that do not contain detailed information (Reyna & Brainerd, 1995; Wolfe, Reyna, & Brainerd, 2005). According to fuzzy trace theory, transfer fails when learners have formed detail-rich, verbatim knowledge representations. Because concrete instantiations communicate an abundance of extraneous information in comparison to more generic instantiations, presenting a learner with a concrete instantiation may encourage the inclusion of extraneous details in the representation, which hinders subsequent transfer.

As mentioned previously, one way of highlighting relational structure and improving transfer is through learning multiple instantiations of the same concept, particularly when learners compare instantiations. Given that a generic instantiation allows the learner to spontaneously recognize and transfer structure, we hypothesized that learning a generic instantiation may be a more efficient route to a representation that allows for successful transfer than learning multiple concrete instantiations. We tested this hypothesis by assigning learners to conditions in which they learned one, two, or three concrete instantiations or one generic instantiation (Kaminski, Sloutsky, & Heckler, 2008). We used the previously described concrete and generic instantiations and two other concrete instantiations. The two additional concrete instantiations involved pizzas and tennis balls and were designed, as was the measuring cup scenario, to tap everyday knowledge in familiar contexts. We equated the amount of training and testing across condition; all participants were presented with the same rules and the same number of examples, questions with feedback, and test questions. After learning, participants were given the same transfer task used in our previous experiments. We found a clear transfer advantage for learning a single generic instantiation. Participants in the generic condition had approximately 78% accuracy, while participants in the concrete conditions scored little or no better than chance, at 38%.

In two follow-up experiments, we attempted to highlight structure between learned instantiations (Kaminski et al., 2008). First, we considered whether giving participants the correspondence of analogous elements across two concrete learning instantiations would help integrate the representations of each and increase the salience of the common structure. This manipulation resulted in no improvement on transfer; scores were again no different than chance. Second, we asked participants, after they had learned two concrete instantiations, to compare them, match analogous elements, and write down any observed similarities. All participants correctly matched elements, but the distribution of transfer scores was bimodal. Approximately 44% of our participants scored high on the transfer test (mean score of 95% correct). The remaining 56% of participants did not do well (mean score of 51% correct). We concluded that the act of explicit comparison may help some learners transfer but may not help others. Moreover, although those who did transfer well scored very high on their initial test of learning, not all who scored high on learning succeeded in transferring after comparison. Therefore, successful learning is a necessary, but not sufficient, condition for successful transfer after comparison.

Given that concrete instantiations may have an advantage for initial learning and generic instantiations can have an advantage for subsequent transfer, we considered a possible "best of both worlds" scenario. We compared transfer after learning the concrete instantiation and then the generic instantiation to transfer when learning only the generic instantiation. Participants in both conditions successfully transferred, but those who learned only the generic significantly outperformed those who learned both (84% correct vs. 65% correct; Kaminski et al., 2008).

Taken together, these findings suggest that learning a generic instantiation of a mathematical concept can be an efficient, direct route to a schematic knowledge representation that can allow for successful transfer. Relational structure is the salient aspect, while elements and other superficial features can be interchanged with those of other instantiations. On the other hand, the course of forming such a representation from learning concrete instantiations is not as efficient, requiring learning more than one instantiation with less likelihood of subsequent transfer than after learning a single generic. When only one concrete instantiation is learned, superficial information dominates the representation and, in turn, interferes with transfer. Even when two and three concrete instantiations were learned in sequence, transfer failed, thus suggesting that these representations were stored independently of each other and not integrated. The fact that learning a concrete followed by a generic instantiation resulted in less transfer than learning a single generic one suggests that superficial information remained in the representation, interfering with successfully applying structural knowledge to the transfer domain.

THE PROBLEM WITH CONCRETENESS

Why is extraneous information in the learning context so damaging for transfer? We suggest that superficial information diverts attention from the relevant relational structure. Attentional resources are limited, and evidence suggests that superficial features and relational structure may compete for attention (DeLoache, 1991; Goldstone, Medin, & Gentner, 1991; Uttal et al., 1999). Goldstone, Medin, and Gentner (1991) suggested that in making similarity comparisons between two situations, attention is split into two separate pools, one for relational similarities and one for superficial similarities. As one pool gets larger, it pulls attention toward itself and away from the other pool.

For concrete instantiations, the superficial features are salient and attention grabbing. It is possible, then, that attention is allocated to these superficial features and diverted from relational structure, not only during similarity comparisons but also in the formation of representations of conceptual knowledge. When attempting to transfer, the learner needs to distinguish relevant from irrelevant information and inhibit the irrelevant. Generic instantiations have less superficial information and thus permit attention to be focused more easily on relevant relational structure.

The results we have discussed in this chapter involved undergraduate college students. It is possible that college students can successfully learn generic instantiations and transfer structural knowledge, but younger learners may need a concrete instantiation to begin to grasp an abstract concept. However, young children are less able than adults to control their focus of attention (Dempster & Corkill, 1999; Diamond, 2006; Napolitano & Sloutsky, 2004). Therefore, if the difficulty with concrete instantiations is due to extraneous information diverting attention from relevant structure, then concreteness may be at least as detrimental for younger children's transfer as it is for older students. To test this possibility, we taught 11-year-old children either the concrete or the generic instantiation and presented them with the transfer domain, as in our earlier experiments. Participants in both conditions successfully learned, but those who learned the concrete instantiation scored higher than those who learned the generic (82% vs. 66% correct). However, for the learners in the concrete condition, transfer scores were only marginally above the chance score of 38% (47% correct), whereas transfer scores in the generic condition were significantly above chance (61% correct; Kaminski, Sloutsky, & Heckler, 2006b). These results suggest that although the concrete instantiation was easily learned, it created an obstacle for children to align structure and successfully transfer. These findings further support the argument that concrete instantiations hinder transfer because the extraneous information diverts attention from relevant structure.

SUMMARY

Our research has compared learning of a novel mathematical concept through concrete instantiations or through a single generic instantiation. We found that relevant structure can be acquired from either concrete or generic instantiations, but the manner in which it is internally represented by the learner is categorically different in each case. Concrete instantiations communicate abundant extraneous information that may pull attention away from the relevant relational structure. The result is a representation in which the superficial is salient. This salient superficial information obfuscates the analogy between learned and novel isomorphs because the learner is unable to recognize structure in the novel situation. As a result, transfer fails. Successful transfer from concrete instantiations requires additional measures such as directly aligning structure for the learner across instantiations or asking the learner to compare multiple instantiations. However, potential transfer domains are not always known a priori, making direct alignment often impossible, and comparison may not always result in success.

Nevertheless, it seems that relational structure is the salient aspect of representations formed from generic instantiations. Consequently, learners spontaneously recognize structure and successfully transfer. Generic instantiations of mathematics, such as traditional symbolic notation, can be powerful educational tools providing efficient routes to portable knowledge representations. Knowledge gleaned from such instantiations can be applied to analogous situations that may appear on the surface to be quite dissimilar.

DISCUSSION

The appeal of concrete learning material is certainly understandable. Concrete instantiations of mathematical concepts are often perceptually rich and attractive. They can perhaps generate a level of initial engagement and interest for students that generic symbols may not. Some concrete instantiations may be familiar and tap prior knowledge to provide a leg up in the learning process. Yet the very aspects of concrete instantiations that make them engaging may also render them ineffective at promoting transfer. The complete story of how concreteness influences the learning, transfer, reasoning, and problem solving of mathematical knowledge over the lifetime of an individual is likely a complex one. The results of the research discussed here pinpoint some of the difficulties learners encounter with concrete instantiations of novel concepts.

In educational practice, concrete material such as base-10 blocks, Cuisenaire rods, and many real-world instantiations such as pizzas are commonly used. Many may wonder whether very young children may need concrete instantiations to begin to acquire mathematical concepts, such as place value or fractions. However, we are aware of no research that demonstrates an advantage of concrete material over more generic material with respect to transfer. If it is true that the difficulty in transferring mathematical knowledge from concrete instantiations stems from extraneous information diverting attention from the important underlying structure, then we would expect that younger children would also have difficulty with concrete instantiations. The ability to inhibit irrelevant information depends on components of executive function that improve through the course of development (Diamond, 2006). Therefore, we would expect a transfer advantage for generic instantiations over concrete instantiations for younger, preadolescent children as well.

Even so, as Blair and Schwartz discuss in Chapter 4 of this volume, educational learning activities often involve an integration of symbolic mathematics and concrete examples. As they illustrate, it is possible to design activities for some concepts in which students can benefit from interacting with concrete instantiations. Further research is needed into the benefits and costs of concrete material for learning and transfer of abstract concepts.

For pedagogical purposes, the possible advantages of choosing concrete learning material over more generic material need to be weighed carefully against the disadvantages, especially for adolescents and young adults who must acquire abstract concepts. In particular, two important questions should be addressed. First, what is the goal of the educational material at hand? If the goal is to learn a single domain, some concrete contextualization may not be a big obstacle. If the goal is to acquire knowledge that can be applied to a variety of superficially dissimilar situations, then the results of our studies suggest that generic material has a clear advantage. Second, what are the possible options for the learning material? In other words, concrete compared to what? For example, story problems about the acceleration of a thrown baseball are more concrete than analogous, strictly symbolic problems, but less concrete than actually measuring the acceleration of a real ball.

If the goal of learning is to acquire knowledge that can be broadly transferred, then generic instantiations are powerful. For mathematical concepts, an important aim of education is such broad transfer. Mathematics is expected to be successfully applied to many real-world situations. Some of these situations may be foreseeable, such as planning personal finances, and so it is reasonable to include such concrete instantiations in the course of formal learning. However, the manner in which mathematics can be applied to less understood situations is not necessarily foreseeable. For example, this is the challenge faced by many scientists as they venture into previously unexplored areas: to recognize consistent relational structure among elements and to transfer structural knowledge from a known analogous domain or model those situations with mathematical expressions. Those faced with the challenge of understanding the structure of unfamiliar domains may be well equipped by having acquired mathematical knowledge in adolescence and young adulthood through generic instantiations.

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