A transfer advantage of learning diagrammatic representations of mathematics

Jennifer A. Kaminski (jennifer.kaminski@wright.edu)
Department of Mathematics and Statistics, Wright State University
3640 Colonel Glenn Hwy, Dayton, OH 45435 USA

Abstract

This study examined learning and transfer of a simple mathematical concept when learning a symbolic sentential format versus learning a diagrammatic format. Undergraduate college students learned an instantiation of a cyclic group and were then given a test of a novel isomorphic group of the same order followed by a test of a novel non-isomorphic group of a higher order. The results were that both the sentential and the diagrammatic formats led to successful learning and transfer to the novel isomorphic group. However, only learning from the diagrammatic representation produced successful transfer to the non-isomorphic group. These findings suggest that learning a diagrammatic representation of a mathematical concept can have transfer advantages over learning strictly sentential formats.

Keywords: Learning; Transfer; Mathematics; Diagrams.

Introduction

Mathematical concepts are often difficult for students to acquire. Part of this difficulty may be related to the fact that mathematics is generally expressed with abstract symbols, such as variables. Mathematical symbols can be challenging for students to interpret and use, leading to misconceptions and obstacles to learning. For example, many algebra students believe that if x is an integer, then y is the next larger integer (Wagner, 1981, 1983). Another common misconception is that equivalent equations with different variables, such as $7 \times w + 22 = 109$ and $7 \times n + 22 = 109$, have different solutions (Wagner, 1981, 1983).

Other evidence for the difficulty of using symbols comes from comparing performance on purely symbolic tasks to analogous contextualized tasks and finding an advantage for reasoning and problem solving in the contextualized formats (e.g. Saxe, 1988; Koedinger & Nathan, 2004; Koedinger, Alibali, & Nathan, 2008). For example, students are frequently more successful solving simple algebra problems when presented as story problems than when presented as symbolic expressions (Koedinger & Nathan, 2004; Koedinger, et al, 2008). The advantage of contextualized situations may be that when contexts are familiar to students, they can derivate mathematical structure from the context itself (Bassok, 1996, 2003). For instance, given a situation involving 12 tulips and 3 vases, students tend to divide 12 by 3 instead of performing another arithmetic operation because a group of flowers is typically divided between a number of vases. Familiar contextualization may also facilitate learning of new concepts (e.g. Kaminski, Sloutsky, & Heckler, 2013).

Although contextualized representations of mathematics may sometimes facilitate reasoning, problem solving, and initial learning, such representations can hinder transfer of mathematical knowledge to novel situations (Kaminski, Sloutsky, & Heckler, 2008, 2013). When college students learned an algebraic system through a familiar context that facilitated initial learning, they were unable to transfer knowledge to a novel analogous domain. However, students who learned the same concept through a generic symbolic format successfully transferred knowledge. Transfer failure may be due to the fact that contextualized real-world instantiations of mathematics communicate more nonessential information than simple symbolic instantiations (Kaminski et al, 2013). This nonessential information is often salient and may divert attention from the less salient mathematical structure, making it difficult to recognize the mathematical structure in novel, superficially dissimilar situations (Kaminski, et al, 2008, 2011, 2013).

However, an important question remains. Does all extraneous information hinder transfer? Perhaps some representations of mathematics have extraneous information that can facilitate transfer. One possible type of representation is a visual display that helps communicate the relevant global relational structure. Such displays are diagrams, and include graphs, matrices, tables, as well as some nonstandard representations. These visual displays instantiate a system with minimal extraneous information, but contain perceptual information that helps spatially organize the elements of the system according to the relevant relational structure. Previous research has demonstrated that diagrams may have advantages over sentential representations for reasoning and problem solving (e.g. Cheng, 2002; Lakin & Simon, 1987), analogical transfer (e.g. Gick & Holyoak , 1983; Pedone, Hummel, & Holyoak, 2001), and non-isomorphic transfer (Novick & Hmelo, 1994).

The advantage of an effective diagram over a sentential representation may be increased salience of the relations between elements and an ability to accommodate a different number of relevant elements. Consider the example of probability. A sentential format would list relevant probabilities of events A, B, C, as $P(A), P(B), P(A|C)$, etc. A tree diagram could visually highlight the relationships between events and could be modified to include additional events. By doing so, diagrams may help to communicate higher-order structure, which may allow the learner to transfer knowledge not only to isomorphic situations (i.e. structurally analogous situations) but also to non-isomorphic situations of the same structural class.

For pedagogical reasons, it is important to examine conditions that promote both isomorphic and non-isomorphic transfer because application of mathematical knowledge involves both isomorphic transfer (e.g. transfer of solution strategies to analogous story problems) as well as non-isomorphic transfer (e.g. solution techniques for systems of two variables applied to systems of more than two variables). From a theoretical perspective, it is important to understand how both types of transfer processes are related. Many theories of analogical transfer posit that successful transfer requires alignment of structure across a familiar domain and an isomorphic target domain; this alignment places analogous elements in a one-to-one correspondence across domains (e.g. Gentner, 1983, 1988; Gentner & Holyoak, 1997; Holyoak & Thagard, 1989, 1997). From multiple instances, learners may form abstract schematic representations that reflect commonalities of the instances (Gick & Holyoak, 1983; Novick & Holyoak, 1991; Reed, 1993). These theories can account for transfer of mathematical knowledge across isomorphs. However, it is unclear how they can account for transfer across non-isomorphic domains in which structure cannot be aligned across instances. If learning of diagrams allows transfer to non-isomorphic situations, do learners align structure when transferring to an isomorphic situation?

The goal of the present research was to examine learning and transfer of a novel mathematical concept from a strictly sentential symbolic representation versus a diagrammatic representation. This study examined both isomorphic transfer (transfer to another system with the same relevant structure and the same number of elements) and non-isomorphic transfer (transfer to another system with the same relevant structure but a different number of elements). When learning the strictly sentential representation, participants may acquire only knowledge of isolated relations between elements and may not gain insight into the higher-order mathematical structure. When learning the diagrammatic representation, participants may learn more than isolated relations between individual elements. They may acquire a structural representation of the concept that can be modified to include more elements than initially learned. As such, the diagram may help communicate higher-order structure, and learning this diagram may facilitate recognition of this structure in novel isomorphic domains as well as non-isomorphic domains of the same type of structure. Therefore, it is hypothesized that both the sentential and diagrammatic representations will result in successful learning and isomorphic transfer, but only the diagrammatic representation will result in successful non-isomorphic transfer.

The concept under consideration was that of a cyclic group (defined in the Method section). Participants learned an instantiation of a cyclic group of order 3 (i.e. three unique elements) with or without the inclusion of a diagram. Participants were then tested on a novel cyclic group of order 3 to examine isomorphic transfer. They were also asked to match analogous elements across domains to investigate whether there are differences in structural alignment when learning a strictly sentential versus a diagrammatic representation. Afterward participants were tested on a novel cyclic group of order 4 (i.e. four unique elements) to examine non-isomorphic transfer.

Experiment

Method

Participants Fifty-eight undergraduate students from a large Midwestern university participated in the experiment and received partial credit for an introductory psychology course.

Materials and Design The experiment included three phases: (1) training and testing in a learning domain, (2) testing in an isomorphic transfer domain, and (3) testing in a transfer domain of the same structure as the learning domain but higher order. Participants were randomly assigned to one of three conditions (Diagram, No Diagram, or Baseline). Participants in the Diagram and No Diagram conditions learned different instantiations of a cyclic group of order 3 during the first phase of the experiment. Participants in the Baseline condition proceeded directly to phase 3, omitting phases 1 and 2. The purpose for the Baseline condition was to measure spontaneous performance in the non-isomorphic transfer domain, without prior instruction on the concept. The isomorphic transfer domain (phase 2) was used in several previous studies (Kaminski, et al, 2008, 2013); without first learning an isomorphic domain, participants were unable to score above chance on this transfer domain.

The learning domain and two transfer domains were artificially constructed instantiations of the concept of a cyclic group. The learning domains and the first transfer domain were of order 3 (i.e. had three unique elements), and the second transfer domain was of order 4 (i.e. had four unique elements). A *Cyclic Group of Order n* is a set of n elements, or equivalence classes, and an associated binary operation over which the following algebraic properties hold: associativity, commutativity, existence of identity, and existence of inverses. This means that if the operation is denoted by "+", then the following are true. The *Associative Property* states that for any elements, x, y, z, of the set, $(x + y) + z = x + (y + z)$. The *Commutative Property* states that for any elements x, y of the set, $x + y = y + x$. Also, there is an element, I, in the set called the *Identity Element*, such that for any element, x, $x + I = x$. Finally, for any element, x, there exists an *Inverse Element*, y, such that $x + y = I$. In addition, a cyclic group is a group that can be generated by a single element. This concept is equivalent to addition modulo n.

The concept of a cyclic group can be instantiated in an unlimited number of ways. The instantiations used for both the Diagram and No Diagram conditions involved three arbitrary symbols, ●, ◆, and ⚑. Participants learned the principles of a cyclic group instantiated as associations between the symbols. The difference between the conditions was the presence or absence of a diagram, the procedure for using the diagram, and the associated cover stories.

In the No Diagram condition, the instantiation was described to participants as rules of a symbolic language in which combinations of two or more symbols yield a predictable resulting symbol. Statements were expressed as *symbol 1 symbol 2 → resulting symbol*. Table 1 shows the symbols, the specific rules, and examples. In the Diagram condition, the cyclic group was described to participants as rules for a code-breaking device that can be used to decode sequences of symbols. The decoding device appeared as a circle with three equally spaced positions marked. One symbol was placed at each position. Given a sequence of two or more symbols, the decoder could be used to determine a resulting symbol by starting at the first symbol and moving clockwise around the dial shown in Figure 1. Figure 1 also presents the procedure for using the decoder, the specific rules, and an example.

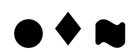
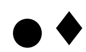
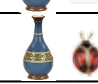
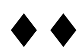
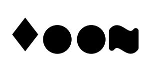
In both conditions, participants were taught the same associations between sequences of symbols and saw the same sentential statements, *symbol 1 symbol 2 → resulting symbol*. The rules, examples, and test questions were identical in both conditions. Aside from different cover stories, the only difference between the conditions was the inclusion of the diagrammatic representation (i.e. the decoding device) and its associated procedure in the Diagram condition. At the end of phase 1, participants were tested with a 24-question multiple-choice test.

The second phase of the experiment was testing of an isomorphic transfer domain. This transfer domain was identical for both conditions and was also a cyclic group of order 3 involving three images of perceptually rich objects. It was described as a children's game where children sequentially point to objects and "the winner" points to a final object (see Transfer Domain 1 in Table 1). Participants were told that the correct final object is specified by the rules of the game (which were the rules of a cyclic group). Furthermore, in both conditions they were told that the rules were like those of the system they just learned. No explicit training in the transfer domain was given; instead, participants were shown a series of examples from which the rules could be deduced (see operands and results for Transfer Domain 1 in Table 1). Participants were asked to figure out the rules of the game by using their knowledge of the learned system. Then they were tested with a 24-question multiple-choice test, isomorphic to the test in the learning phase of the Diagram and No Diagram conditions, but using the elements of the transfer domain.

Following the test, participants were asked to match analogous elements across the learning and transfer domains. Correct matching of elements was taken as an indicator of correct structural alignment between the learning and transfer domains. For cyclic groups of order 3, there are two possible correct mappings between groups. The identity element is unique; therefore a correct mapping must align these two elements across domains. However, the mapping between remaining two elements is not unique. Therefore, a response was considered correct if (a) the mapping was *one-to-one* and *onto* (i.e., each learning element corresponded to a single transfer element and each element of the transfer elements were used) and (b) the mapping preserved the identity element. In other words, if a participant used each of the group elements and mapped the identity element correctly, then the response was correct. Because the critical aspect was correctly choosing the identity element and most participants were expected to form mappings that were *onto*, 33% accuracy was used as a conservative measure of chance for a group of participants.

The third phase of the experiment was testing of a non-isomorphic domain of the same structure as the learning domain (i.e. a cyclic group of order 4). Participants were given a paper and pencil ten-question multiple-choice test (see Table 2) and told that the knowledge of the system they learned first can help them figure out the new system. They were also given five example statements (the operands and results shown in Table 2) from which the complete set of rules could be deduced.

Table 1: Stimuli for the learning and isomorphic transfer domains.

	Learning Domain Cyclic group of order 3		**Transfer Domain 1** Cyclic group of order 3	
Elements				
Identity				
Associations between elements	*(Presented as rules)*		*(Presented as examples)*	
	Operands	Result	Operands	Result
				
				
				
				
Example Test Question	Find the resulting symbol.  → ___ *Answer:* 		If children pointed to these object,  , what object did the winner point to? *Answer:* 	

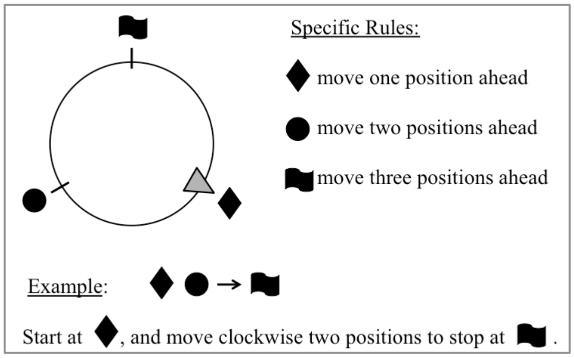


Figure 1: Diagram used in the Diagram Condition

Table 2: Stimuli for non-isomorphic transfer domain.

	Transfer Domain 2 Cyclic group of order 4	
Elements	🌢 ★ ⊙ □	
Identity	🌢	
Associations between elements	*(Presented as examples)*	
	Operands	Result
	🌢 , ★	★
	🌢 , ⊙	⊙
	★ , ★	⊙
	⊙ , ⊙	🌢
	★ , ⊙	□
Example Test Question	Find the resulting symbol. 🌢, ★, ⊙,□ → ? *Answer:* ⊙	

Procedure Participants were seen individually in a lab on campus. Phases 1 and 2 were presented on a computer. Participants proceeded at their own pace, with their responses recorded by the computer. The learning phase consisted of approximately 80 slides and required approximately 15 minutes to complete. The transfer phase (phase 2) consisted of 48 slides and took on average 10 minutes to complete. The second transfer phase (phase 3) took participants approximately 8 minutes to complete. All material in phase 3 was presented on paper.

Results

Four participants (two Diagram and two No Diagram) were removed from the analysis because they failed to learn

Table 3. Mean accuracy (percent correct) on learning and isomorphic transfer. Note: Standard deviations are presented in parentheses. Chance performance is 37.5%

	Learning	Transfer
No Diagram	78.9 (14.5)	78.5 (17.8)
Diagram	85.2 (8.95)	74.5 (22.1)

the concept in phase 1; their learning scores were less than 11 and no different than the chance score of 9 (i.e. 37.5%).

Note that for the following analyses, learning scores and isomorphic transfer scores were not normally distributed in the Diagram condition; the distributions were negatively skewed (SWs <.89, dfs = 18, ps < .04). In addition, non-isomorphic transfer test scores had bimodal distributions in both conditions. Therefore, non-parametric analyses were done to examine each of these scores.

Participants in both conditions successfully learned the concept (see Table 3). Learning scores were above chance in both conditions, Wilcoxon Signed Ranks Test, Zs > 3.72, ps < .001. There were no significant differences in learning levels between conditions, Mann-Whitney U test, U = 123.5, p = .22.

In phase 2 (i.e. testing of an isomorphic transfer domain), participants also performed well in both conditions (see Table 3). Scores were above a chance score of 37.5%, in both conditions, Wilcoxon Signed Ranks Test, Z s > 3.63, ps < .001. Note that previous research demonstrated that without initially learning an isomorphic domain, participants were unlikely to score above chance on this transfer task (Kaminski, et al, 2008, 2013). Therefore, it appears that participants in both conditions successfully transferred structural knowledge acquired in phase 1 to answer questions about the isomorphic domain in phase 2. No significant difference in transfer scores between the two conditions was found, Mann-Whitney U test, U = 151.0, p = .74.

In addition, most participants in both conditions accurately matched analogous elements across the learning and isomorphic transfer domains (83% in the Diagram condition and 78% in the No Diagram condition), suggesting that they successfully aligned analogous structure across the two domains. The percent of participants in both conditions was well above chance of 33% and not different between conditions, Fisher Exact test, ps = 1.00.

While there were no differences in performance levels between conditions for phases 1 and 2, there were significant differences in performance in phase 3 (i.e. testing on a non-isomorphic domain of similar structure). Scores in the Diagram and No Diagram conditions had bimodal distributions; Table 4 presents the frequency of scores at different levels. Transfer scores were higher in the Diagram condition (M = 67.8, SD = 9.69) than in the No Diagram condition (M = 41.1, SD = 7.79), Mann-Whitney U test, U =

Table 4. Percent of participants in each condition scoring at different levels on the non-isomorphic transfer task.

	Accuracy Level		
	Low (0-40%)	Middle (50-70%)	High (80-100%)
Baseline	61	33	6
No Diagram	61	17	22
Diagram	39	0	61

97.5, $p < .04$. Furthermore, scores in the Diagram condition were higher than scores in the Baseline condition (M = 41.1%, SD = 19.1%), Mann-Whitney U test, $U = 107.0$, $p < .04$, one-tailed. However, scores in the No Diagram condition were not significantly different than those in the Baseline, $U = 144.5$, $p = .58$. This finding suggests that the majority of participants in the Diagram condition were able to transfer knowledge of the cyclic group order 3 to the non-isomorphic cyclic group of order 4, but the majority of participants in the No Diagram condition were not able to do so.

Discussion

The goal of the present study was to investigate transfer of mathematical knowledge when learning a strictly sentential representation versus learning a diagrammatic representation. This study considered transfer to a novel isomorphic domain as well as transfer to a novel non-isomorphic domain of the same structural class. Both formats resulted in equally successful learning and isomorphic transfer. However, participants who learned the diagram were more successful at non-isomorphic transfer than those who learned only the sentential format. These findings suggest that although the diagram added non-essential information, this information did not hinder learning or isomorphic transfer. Moreover, the inclusion of this information resulted in a clear advantage for non-isomorphic transfer.

Previous research has demonstrated that learning instantiations that include extraneous information hinder transfer of mathematical knowledge to novel isomorphs because the extraneous information is generally salient and likely diverts attention from the relevant structure (Kaminski et al, 2008, 2011, 2013). Compared to strictly sentential representations, diagrams also communicate nonessential information to the learner. For example, in the present study, it is not necessary to include the diagram; the same rules and associations were learned equally well in the No Diagram condition. Clearly standard diagrams such as tree diagrams, matrices, and graphs also communicate nonessential information. However while the information added by a diagram is nonessential, it is not necessarily irrelevant. Effective diagrams use visual information to spatially organize elements of a system in a way that highlights relations and relevant structure and does not divert attention from the structure. Such diagrams may help to communicate global structure of the system in manner that can be modified if necessary to incorporate a different number of elements.

In the current study, the diagram was circular and likely helped to communicate the cyclic nature of the group and the fact that any element can be obtained as a result of operations involving the other elements. It is more difficult to recognize the cyclic nature of the relationship between elements in the strictly sentential format. Even if learners had constructed a schematic representation of the concept from the sentential format without the diagram, the schema appears to reflect only local associations between three elements and not a more global structure that can be modified and applied to non-isomorphic situations.

With regard to structural alignment, participants in both conditions were equally accurate at matching analogous elements, possibly suggesting structural alignment. However it is not clear that this element-level matching is necessary when learning the diagram or whether global structure can be mapped from learned to target domains without one-to-one correspondence of elements.

Successful non-isomorphic transfer from learning the diagram suggests that participants have formed a more sophisticated internal representation of a structural class of mathematical entities, in this case cyclic groups of different orders. Recognizing that different mathematical entities can fall into the same structural categories is an important part of advancing mathematical knowledge. For example, algebra students should be able to modify techniques for solving systems of equations of two variables to solve systems of equations of more than two variables. Similarly, college students should recognize that slope of a line is an instance of derivative of a function. An effective diagram, if available, may help illuminate structure in a way that allows modification of the number of elements. Standard mathematical diagrams such as matrices, graphs, networks, and Venn Diagrams do precisely this.

At the same time, there may be limitations to the benefit of diagrams. The inclusion of diagrams may not always facilitate initial learning. Correct interpretation and use of diagrams requires additional learning beyond learning standard sentential representations. For some combinations of concepts, diagrams, and learners, such as those considered in this study, a diagram is easily learned. However, this is not always the case. For example, in middle school students, diagrams provided a benefit for solving algebraic word problems only for older students and high-achieving students, but not for younger students and lower-achieving students (Booth & Koedinger, 2012). Some concepts may be simple enough to be learned without such additional representations. For more difficult concepts, some learners may be unable to fully learn the diagram and the relationship between the diagram and standard sentential formats such as equations.

It is also important to note that while a diagram is a visual representation of the elements and relations of a system, it is meaningless without knowledge of how to interpret it. The diagram used in the present study involved the visual representation in Figure 1 along with the procedure of how to use it. The same is true for common mathematical diagrams such as multiplication tables and Cartesian graphs. These well known diagrams easily communicate information to us only because we have been explicitly taught procedures for constructing and interpreting them.

Learning diagrams in addition to standard sentential mathematics may require additional effort. For some learners and some diagrams, this may be challenging. However, the benefit of well-designed diagrams is likely worth the effort. Once learned, diagrams likely can provide advantages for transfer to isomorphic situations and many non-isomorphic situations.

Acknowledgments

This research was supported by a grant from the Institute of Education Sciences, U.S. Department of Education (#R305A140214).

References

Bassok, M. (1996). Using content to interpret structure: Effects on analogical transfer. *Current Directions in Psychological Science,* 5, 54-58.

Bassok, M. (2003). Analogical transfer in problem solving. In J. E. Davidson & R. J. Sternberg (Eds.) *The psychology of problem solving* (pp. 343-369). New York: Cambridge University Press.

Booth, J. L, & Koedinger, K. R. (2012). Are diagrams always helpful tools? Developmental and individual differences in the effect of presentation format on student problem solving. *British Journal of Educational Psychology*, 82, 492–511

Cheng, P. C. H. (2002). Electrifying diagrams for learning: principles for complex representational systems, *Cognitive Science*, 26, 685–736.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science, 7,* 155–170. doi:10.1207/s15516709cog0702_3

Gentner, D. (1988). Metaphor as structure mapping: The relational shift. *Child Development, 59,* 47–59. doi:10.2307/1130388

Gentner, D., & Holyoak, K. J. (1997). Reasoning and learning by analogy. *American Psychologist, 52,* 32–34. doi:10.1037/0003-066X.52.1.32

Gick, M. L., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology, 15,* 1–38. doi:10.1016/0010-0285(83)90002-6

Herscovics, N., & Kieran, C. (1980). Constructing meaning for the concept of equation. *The Mathematics Teacher, 73*(8), 572-580.

Holyoak, K. J., & Thagard, P. (1989). Analogical mapping by constraint satisfaction. *American Psychologist, 52*, 35-44.

Holyoak, K. J., & Thagard, P. (1997). The analogical mind. *American Psychologist, 52*, 35-44.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science, 320*, 454-455.

Kaminski, J. A., & Sloutsky, V. M. (2011). Representation and transfer of abstract mathematical concepts in adolescence and young adulthood. In V. Reyna (Ed.) The Adolescent Brain: Learning, Reasoning, and Decision Making (pp. 67-93). Washington, DC: APA.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2013). The Cost of Concreteness: The Effect of Nonessential Information on Analogical Transfer. *Journal of Experimental Psychology: Applied, 19,* 14-29.

Koedinger, K. R., Alibali, M. W., & Nathan, M. M. (2008). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science, 32*(2), 366-397. http://dx.doi.org/10.1080/03640210701863933.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: effects of representations on quantitative reasoning. *Journal of the Learning Sciences, 13*, 129-164.

Larkin, J. H., & Simon, H. A. (1987). Why a diagram is (sometimes) worth then thousand words. *Cognitive Science*, 11, 65-100.

Novick, L. R., & Holyoak, K. J. (1991). Mathematical problem solving by analogy. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 17,* 398-415.

Novick, L., & Hmelo, C. E. (1994). Transferring symbolic representations across nonisomorphic problems. *Journal of Experimental Psychology: Learning. Memory, and Cognition, 20*, 1296-1321.

Pedone, R., Hummel, J. E., & Holyoak, K. J. (2001). The use of diagrams in analogical problem solving. *Memory and Cognition*, *29*, 214-221.

Reed, S. K. (1993). A schema-based theory of transfer. In D. K. Detterman & R. J. Sternberg (Eds.) Transfer on Trial: Intelligence, Cognition, and Instruction. Norwood, NJ: Alex Publishing.

Saxe, G. B. (1988). The mathematics of child street vendors. *Child Development, 59*, 1415-1425.

Wagner, S. (1981). Conservation of equation and function under transformations of variable. *Journal for Research in Mathematics Education*, 12, 107-118.

Wagner, S. (1983). What Are These Things Called Variables? *The Mathematics Teacher, 76*(7), 474-479.